Quenched Randomness at First-Order Transitions[†]

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Abstract

A rigorous theorem due to Aizenman and Wehr asserts that there can be no latent heat in a two-dimensional system with quenched random impurities. We examine this result, and its possible extensions to higher dimensions, in the context of several models. For systems whose pure versions undergo a strong first-order transition, we show that there is an asymptotically exact mapping to the random field Ising model, at the level of the interface between the ordered and disordered phases. This provides a physical explanation for the above result and also implies a correspondence between the problems in higher dimensions, including scaling relations between their exponents. The particular example of the q-state Potts model in two dimensions has been considered in detail by various authors and we review the numerical results obtained for this case. Turning to weak, fluctuation-driven first-order transitions, we describe analytic renormalisation group calculations which show how the continuous nature of the transition is restored by randomness in two dimensions.

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1 Introduction

The effect of quenched random impurities which couple to the local energy density on the critical behaviour of a system whose pure version undergoes a continuous transition is well understood in terms of the Harris criterion [1]: if the specific heat exponent α of the pure system is negative, such impurities do not change the qualitative nature of the transition or its universality class, while in the opposite case there is expected to be different behaviour described by a new random fixed point of the renormalisation group (RG). Given the volume of literature devoted to this subject, it is rather surprising to find relatively few studies of the effects of such impurities on systems whose pure versions undergo a first-order transition, despite the ubiquity of such systems in nature. A naive extension of the Harris criterion, based on the observation that the effective value of α is unity at a first-order transition, might suggest that randomness is always strongly relevant. On the other hand, first-order behaviour is accompanied by a finite correlation length and is commonly assumed to be stable under small perturbations. Following early work by Imry and Wortis [2], Aizenman and Wehr [3] and Hui and Berker [4] showed that the role of dimensionality d is crucial. In fact the first authors proved a rigorous theorem which states that for $d \leq 2$ the Gibbs state is always unique, for arbitrarily small but non-zero concentration of impurities. This means that there can be no phase coexistence at the transition and hence no latent heat.

This result raises a number of important questions: (a) what is its physical mechanism; (b) is the continuous transition accompanied by a divergent correlation length; (c) if so, what are the (presumably) universal critical exponents characterising the transition; and (d) what happens for $d > 2$? Initial Monte Carlo studies of (c) [5, 6] suggested that the critical behaviour of all such systems might be strongly universal, exhibiting Ising-like exponents independent of the underlying symmetry of the model. This received

some theoretical support [7]. However, more recent numerical studies (to be described below) support earlier theoretical results [8] in suggesting that the critical behaviour of, for example, the random bond q-state Potts model depends continuously on q, both in the region where the pure model has a continuous transition, and where it is first-order. The aim of this talk is to summarise some of these recent developments, and to provide at least partial answers to the above questions.

2   Mapping to the random field Ising model

First, I want to describe a mapping [9] of this problem to the *random field* Ising model (RFIM), which is asymptotically valid for very strong first-order transitions, and which provides a physical explanation of the Aizenman-Wehr result [3], as well as a prediction for what happens when $d > 2$.

Consider a pure system at a thermal first-order transition point. There will be coexistence between a (generally unique) disordered phase and the (generally non-unique) ordered phases. The internal energies U_1 and U_2 of these two kinds of phase will differ by the latent heat, L. Now consider a interface between the disordered phase and one of the ordered phases, with surface tension σ (measured in units of kT_c). If σ is large, there will be very few isolated bubbles of the opposite phase above or below the main interface. Its equilibrium statistics may therefore be described by a free energy functional equal to its area multiplied by σ. Let us compare this with the interface between the two *ordered* phases of an Ising model, with spins taking the values ± 1, at low temperatures. Once again, there will be few bubbles of the opposite phase, and the bare interfacial tension will be $\sim 2J$, where J is the reduced exchange coupling. In the limit when $\sigma \sim 2J$ is large, these interfacial models are therefore identical[1]

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[1]This is not the case when bubbles of the opposite phase are included. For example, regions of ordered phase appearing in the disordered phase are counted with the degeneracy

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Now consider the effects of randomness in these two models, in the first case random bonds, coupling to the local energy density, and in the second random fields, coupling to the local magnetisation. In the RFIM, these are accounted for by adding a term $\Sigma_{r>}h(r) - \Sigma_{r<}h(r)$, where $h(r)$ is the local random field, and in the two terms r is summed respectively above and below the instantaneous position of the interface. For random impurities of local concentration $\delta x(r)$ we have, similarly, $U_1\Sigma_{r>}\delta x(r) + U_2\Sigma_{r<}\delta x(r)$. Apart from a constant independent of the position of the interface, this may be written as L times $\Sigma_{r>}\delta x(r) - \Sigma_{r<}\delta x(r)$, exactly the same form as for the RFIM.

We may therefore set up a dictionary between these two cases, in which the thermal variables of the random bond system are related to the magnetic variables of the RFIM:

$$
\begin{aligned}
\sigma/kT_c &\longleftrightarrow J/kT \\
(L/kT_c)\,x &\longleftrightarrow h_{RF}/kT \\
(T - T_c)\,L &\longleftrightarrow H \cdot M,
\end{aligned}
$$

where the last relation is between the fields $(T - T_c)$ and a *uniform* magnetic field H which respectively distinguish between the two phases. Although the above mapping may seem ill-defined in its use of the local energy density as a kind of order parameter, it may be made completely explicit, for example, for the q-state Potts model through the mapping to the random cluster model, where $\sigma \sim L \sim \ln q$ for large q [9].

The interfacial model of the RFIM has been studied extensively [10] and, in particular, RG equations have been derived which are asymptotically exact in the limit of low temperatures and weak randomness, just where our mapping is valid. For $d = 2$ the variable $h_{RF}/J \sim xL/\sigma$ is (marginally) relevant, and, just as this destroys the spontaneous magnetisation for the RFIM, the

factor of the ordered phases, as compared with a factor of unity for the Ising case.
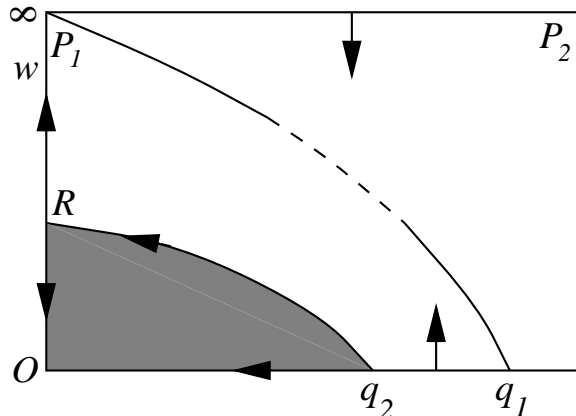
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Figure 1: Critical surface for $d > 2$, constructed by analogy with the phase diagram of the RFIM. The axes are $(1/\sigma, x)$ in the notation of this talk. The shaded region is first-order, bounded by a line of tricritical points, along which the RG flows go towards the fixed point at R. The upper boundary is the percolation limit, and the partially dashed curve is a conjectured line of fixed points describing the continuous random critical behaviour.

latent heat vanishes for the random bond case, in accord with the Aizenman-Wehr theorem [3]. For the RFIM, the RG trajectories flow towards a paramagnetic fixed point at which the correlation length $\xi \sim \exp(\text{const}(J/h_{RF})^3)$ is finite, but note that this is outside the region where the mapping to the random bond problem is valid. We cannot conclude, therefore, anything about the nature of the latter's true critical behaviour from this argument. In fact, numerical and other analytic studies indicate that the actual correlation length is divergent, and hence ξ is merely a crossover length in this case.

The predictions are more interesting for $d > 2$, when the RFIM exhibits a critical fixed point at $kT/J = 0$ and a finite value of h_{RF}/J. As shown in Fig. 1, there is now a region in which the spontaneous magnetisation (latent heat) is non-vanishing as the sign of the uniform field $(T - T_c)$ is changed. This region is bounded, in the case of the RFIM by the critical curve, close

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to which the critical behaviour is determined by the zero-temperature fixed point. Similarly, the first-order region of the random bond system will be bounded by a line of *tricritical* points, above which (presumably) the transition becomes continuous. Since the fixed point occurs in the region where the mapping between the two systems is valid, we may infer some of the tricritical exponents from those of the RFIM [9]. For example, the latent heat should vanish as $(x_c - x)^\beta$, where β is the usual magnetisation exponent of the RFIM. Similarly, the correlation length on the critical surface should diverge as $x \to x_c$ with the usual exponent ν of the RFIM. But the behaviour for $T \neq T_c$ is related to the *magnetic* properties of the RFIM, and is complicated by the fact that the temperature at the RFIM fixed point is dangerously irrelevant, with an RG eigenvalue $-\theta$ which is responsible for the violation of hyperscaling $\alpha = 2 - (d - \theta)\nu$ in that model. As a result, for example, the correlation length in the random bond model at $x = x_c$ diverges as $(T - T_c)^{1/y}$, with $y = d - \theta - \beta/\nu$.

3   Results for the Potts model.

As is well-known, the pure q-state Potts model undergoes a first-order transition for $q > 4$ in $d = 2$, and it is a relatively simple system to study numerically in the random bond case. By choosing a suitable distribution of randomness, one can fix the model to be self-dual so that the critical point is determined exactly. One method, which is very effective for pure two-dimensional critical system, is to study the finite-size scaling behaviour of the eigenvalues Λ_i of the transfer matrix in a strip of width N. By conformal invariance [11], these are related to the scaling dimensions x_i by $2\pi x_i/N \sim \ln(\Lambda_0/\Lambda_i)$ as $N \to \infty$. For the Potts model, it is possible [12] to write the transfer matrix in the so-called connectivity basis [13], allowing q to appear as an easily tunable variable. In the random case, however, the transfer matrices for different rows do not commute, and the role of the Λ_i is

taken by the Lyapunov exponents which govern the average growth rate of the norm of vectors under the action of the transfer matrices. The asymptotic behaviour of the correlation functions is related to these exponents as for the pure case. In order to use conformal invariance, it is necessary to assume translational invariance which is only recovered after quenched averaging. However, the Lyapunov exponents themselves are not self-averaging, only their logarithms. This is related to the phenomenon of multi-scaling, whereby the average of the pth power of the correlation function is governed by an exponent $x^{(p)}$ which is not linear in p. Since this occurs in most random systems, however, I shall not treat the problem in detail here. It still proves possible, effectively by measuring the whole distribution of the $\ln \Lambda_i$, to extract the decay of the average correlation function, and hence, for example, the magnetic exponent $x_1 \equiv \beta/\nu$. The results from our study [12] are shown in Fig. 2. We see that the value of x_1 appears to increase steadily with q. For $q < 3$ it agrees with the analytic $(q-2)$-expansion [8, 14]. There appears to be no break at $q = 4$ where the pure transition becomes first-order. Our results disagree with earlier Monte Carlo work [5] for $q = 8$ where a number close to the Ising value of $\frac{1}{8}$ was reported. However, more recent Monte Carlo results by Picco [15] find $x_1 = 0.150 - 0.155$ for $q = 8$ (and 0.185 ± 0.005 for $q = 64$, while Chatelain and Berche [16] report $x_1 = 0.153 \pm 0.003$ for $q = 8$. The small discrepancy with our results may be explained by a careful study of the crossover behaviour which shows that stronger randomness (beyond the reach of our methods) must be considered as q increases [15].

All studies report a thermal exponent $\nu \approx 1$. This approximately saturates the lower bound proved by Chayes et al [17]. It has been suggested [18] that, in general, this might be a result of the averaging procedure and that in some random systems the 'true' value of ν might be less than unity. However, it can shown that this is not the case here.
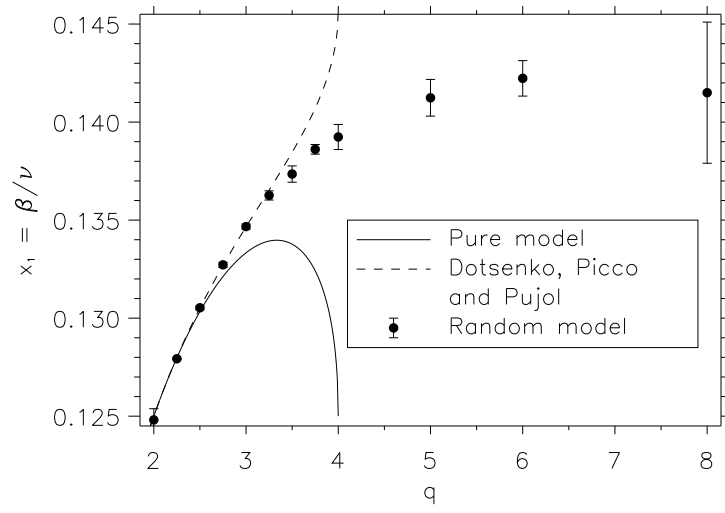
Figure 2: Measured values for the magnetic exponent β/ν of the random-bond q-state Potts model for $d = 2$, from Refs. (9,12). The solid curve is the exact value for the pure model for $q \leq 4$, and the other is the extrapolated $(q - 2)$-expansion of Dotsenko et al. [14].

4 Analytic results for weak first-order transitions.

Although the q-state Potts model is relatively simple to analyse numerically, like most other first-order transitions it is difficult to study in any kind of perturbative RG approach, since the randomness is strongly relevant. However, this is not always the case for systems which exhibit weak 'fluctuation-driven' first-order transitions. An example is afforded by N Ising models with spins $s_i(r)$, coupled through their energy densities, with a hamiltonian

$$\mathcal{H} = -\sum_{r,r'} J(r,r') \sum_i s_i(r)s_i(r') + g \sum_{r,r'} \sum_{i \neq j} s_i(r)s_i(r')s_j(r)s_j(r') \qquad (1)$$

For uniform couplings J, this exhibits a first-order transition when $N > 2$, and in fact, on the critical surface is equivalent to the Gross-Neveu model, which may be analysed nonperturbatively to show that the correlation length is $\xi \sim \exp(\text{const}/(N-2)g)$ [19]. Even with random bonds $J(r,r')$ of strength Δ the one-loop RG equations may be derived By very simple combinatorial methods [20], to give [21]

$$dg/d\ell \ = \ 4(N-2)g^2 - 8g\Delta + \cdots \qquad (2)$$

$$d\Delta/d\ell \ = \ -8\Delta^2 + 8(N-1)g\Delta + \cdots \qquad (3)$$

These flows are illustrated in Fig. 3. When $\Delta = 0$, g runs away to infinity, in a manner consistent with the non-perturbative result for ξ. When randomness is added, however, it is marginally relevant, and in fact the flows are quite similar to those found for the RFIM quoted earlier if we identify the interfacial tension $\sim \xi$. However, in this case, we also see where the flows end: in this case at the critical fixed point corresponding to N decoupled pure Ising models. (This is the only example I know of where the infrared and ultraviolet fixed points of a set of RG flows are the same.) There have been various generalisations of this calculation to coupled Potts models [22]. In most
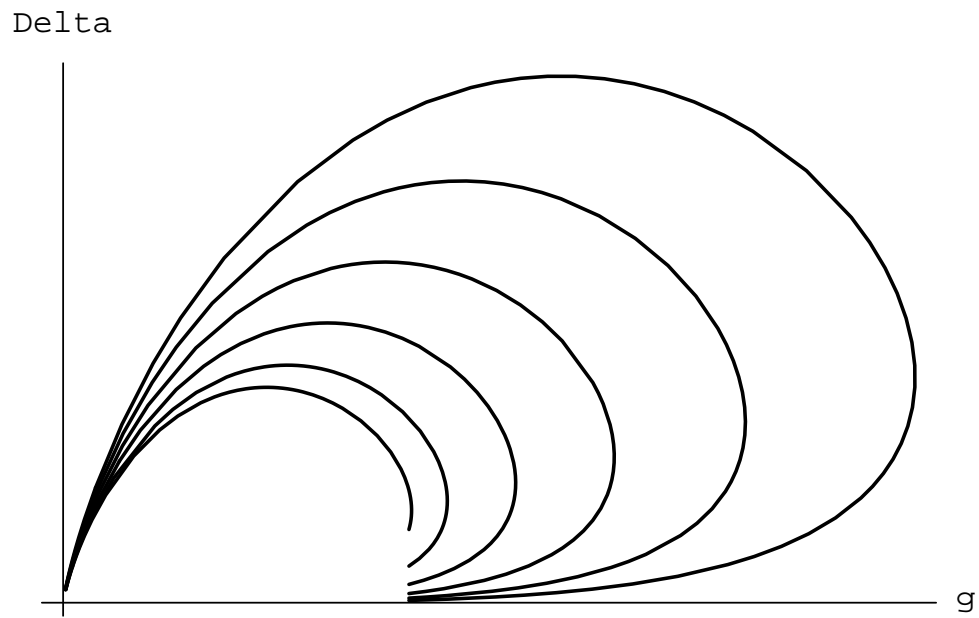
Figure 3: RG flows in the critical surface for N coupled Ising models, from Ref. (21). For non-zero randomness the trajectories curve back towards the decoupled pure fixed point.

cases the addition of randomness induces flow towards a critical fixed point which is perturbatively accessible. In higher dimensions, however, different outcomes are possible. Adding randomness to models in $4 - \epsilon$ dimensions with cubic anisotropy appears not to change their fluctuation-driven first-order character [21]. However, for impure n-component superconductors in $4 - \epsilon$ dimensions there is a critical concentration above which the transition becomes continuous [23].

5 Outlook

There are still many open questions in this relatively little explored area. A lot more needs to be learned about the nature of the universal critical behaviour (if indeed it is universal), both in $d = 2$ and for $d > 2$ when the randomness is sufficiently strong. In the former case, it may be possible to solve the problem by conformal field theory methods. Even the limit of large q appears non-trivial, however. Similar ideas apply to quantum phase transitions and may have relevance for the quantum Hall effect. Most importantly, it should be possible to find experimental systems which realise some of the predictions discussed above. There are, after all, many three-dimensional examples of first-order transitions. However, it should be stressed that the randomness should couple only to the local energy density, not to the order parameter, otherwise this becomes the random field problem. It is also important to ensure that the tricritical behaviour is driven by the effects discussed and not by some simple mean-field mechanism (which would give rise to mean-field tricritical exponents in $d = 3$, part from logs.) Finally it is important to understand the dynamics of these systems: it may be that, like the random field problem, they are plagued by logarithmically slow time scales close to the critical point [24].

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